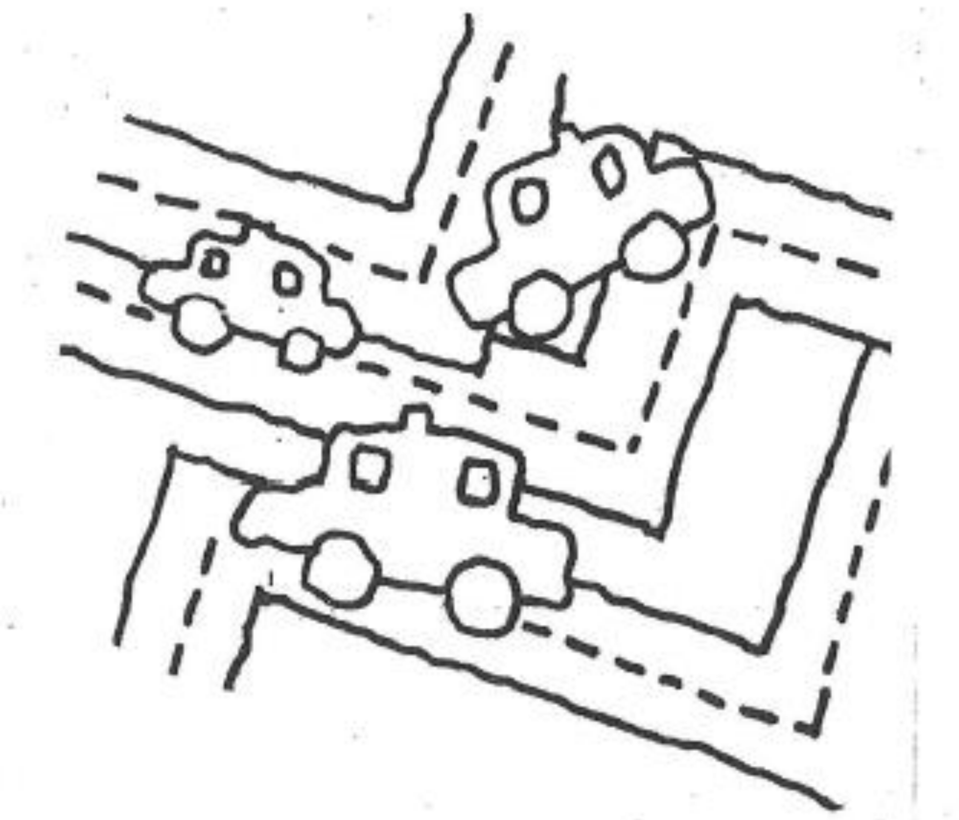
TAXICAB GEOMETRY*

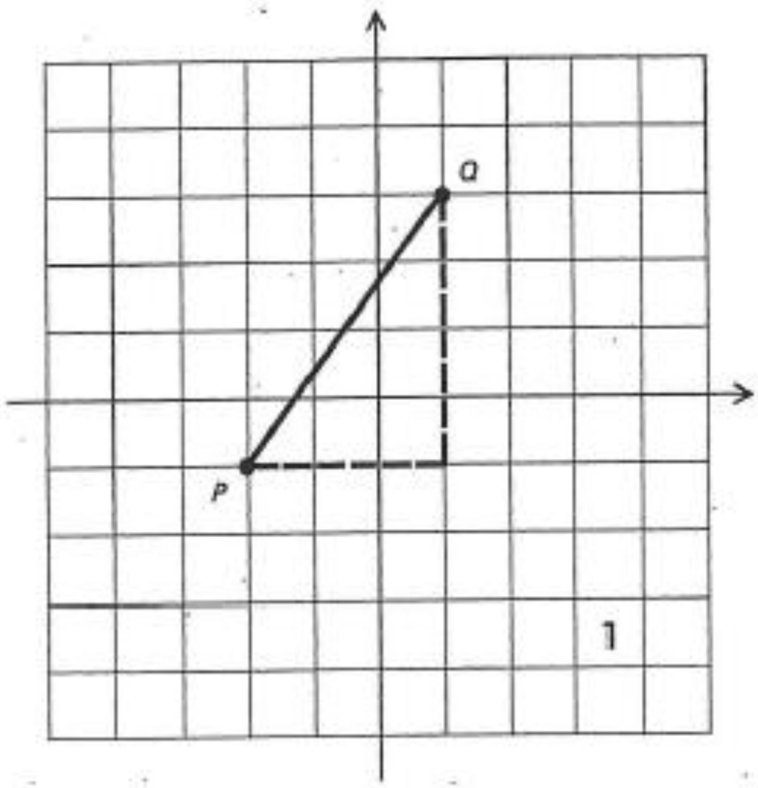
10th Grade Project Week 23-24



*All problems and intros taken from *Taxicab Geometry*, by Eugene F. Krause

WHAT IS TAXICAB GEOMETRY?

The usual way to describe a (plane) geometry is to tell what its points are, what its lines are, how distance is measured, and how angle measure is determined. When you studied Euclidean coordinate geometry the points were the points of a coordanitized plane. Each of these points could be designated either by a capital letter or by an ordered pair of real numbers (the "coordinates" of the point). For example, in Fig. 1, $P = (-2, -1)$ and $Q = (1,3)$. The lines were the usual long, straight, skinny sets of points; angles were measured in degrees with a (perfect) protractor; and distances either were measured "as the crow flies" with a (perfect) ruler or were calculated by means of the Pythagorean Theorem.



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For example, In Fig. 1 the distance from P to Q could be found by considereing a right triangle having $\overline{PQ}$ as its hypotenuse. The dotted segments are the legs of one such right triangle. (Are there any other such right triangles?) These legs clearly have lengths 3 and 4. Thus, by the Pythagorean Theorem, the Euclidean distance from P to Q is $\sqrt{3^2 + 4^2} = 5$. We shall use the symbol d_E to represent the Euclidean distance function. Thus, in our example we would write

$$d_E(P, Q) = 5$$

And read it "The Euclidean distance from P to Q is 5."

Taxicab geometry is very nearly the same as Euclidean coordinate geometry. The points are the same, the lines are the same, and angles are measured in the same way. Only the distance function is different. In Fig. 1, the taxicab distance from P to Q , written $d_T(P,Q)$, is determined not as the crow flies, but instead as a taxicab would drive. We count how many blocks it would have to travel horizontally and vertically to get from P to Q. The dotted segments suggest one taxi route. Clearly

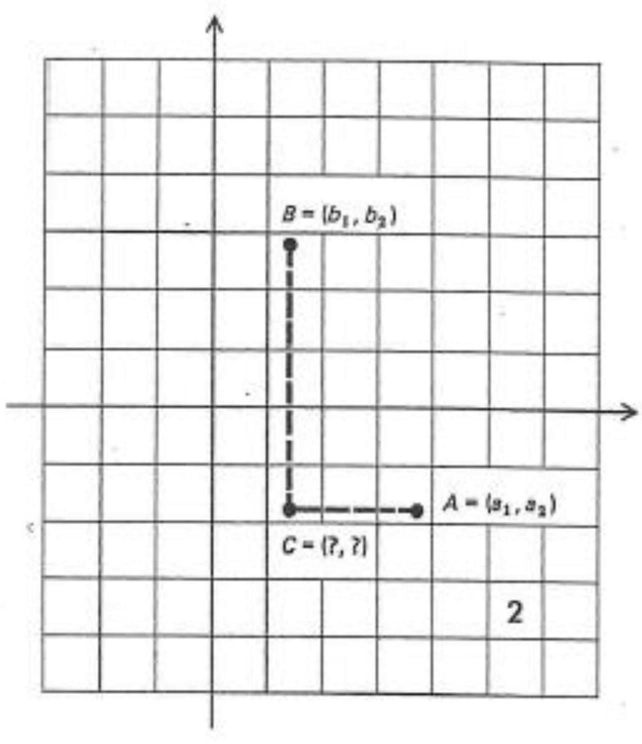
$$d_T(P,Q) = 7.$$

"The taxi distance from P to Q is 7."

Figure 2 is a reminder that most of the points of the coordinate plane do not have two integer coordinates. In the figure a pair of "arbitrary" points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ is given. What are the coordinates of the point C? Write an expression for the length of $\overline{AC}$ in terms of the coordinates of A and C. Write an expression for the length of $\overline{BC}$ in terms of the coordinates of B and C. The following precise, algebraic definitions of d_T and d_E should now seem reasonable:

(1) $d_T(A,B) = |a_1 - b_1| + |a_2 - b_2|$;

(2) $d_E(A,B) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$.



We will make use of these careful definitions only very rarely. The reason for inserting them here is to assure ourselves that (1) there is a mathematically respectable foundation underlying taxicab geometry, and (2) there is a definite taxicab distance between any two points, whether they are located at a "street corner" or not.

SECTION I: STRUCTUED PRACTICE

In Section I, you will do a series of exercises to help get you comfortable in the world of taxicab geometry and to start discovering how some basic properties of geometry change. Look for the **bold** notes, as these are meant to help direct you.

PROBLEM SET 1

1. On a sheet of graph paper, mark each pair of points P and Q and then find both $d_T(P,Q)$ and $d_E(P,Q)$.

    a. $P = (5,4), Q = (1,2)$
    b. $P = (-4,3), Q = (3,2)$
    c. $P = (-5,-4), Q = (1,-2)$
    d. $P = (3,-1), Q = (-2,4)$
    e. $P = (4,-3), Q = (-2,-3)$

2. Given the points your graphed above, answer the following questions.

    a. If $d_T(A,B) = d_T(C,D)$ must $d_E(A,B) = d_E(C,D)$?
    b. If $d_E(A,B) = d_E(C,D)$, must $d_T(A,B) = d_T(C,D)$?
    c. Under what conditions on A and B does $d_T(A,B) = d_E(A,B)$?
    d. Is it always true that $d_E(A,B) \leq d_T(A,B)$? Try to probe your answer using the formal definitions (1) and (2).

3. For this exercise, $A = (-2,-1)$. Mark A on a sheet of graph paper. For each point P below calculate $d_T(P,A)$ and mark P on the graph paper. Graph a-h all on the same graph.

    a. $P = (1,-1)$
    b. $P = (-2,-4)$
    c. $P = (-1,-3)$
    d. $P = (0,-2)$
    e. $P = (\frac{1}{2}, -1\frac{1}{2})$
    f. $P = (-1\frac{1}{2}, -3\frac{1}{2})$
    g. $P = (0,0)$
    h. $P = (-2,2)$

4. Complete the following on the **same graph** as problem 3.

    a. Find some more points at taxi distance 3 from A.
    b. Graph the set of all points P at taxi distance 3 from A; that is, graph $\{P \mid d_T(P,A) = 3\}$.

    The set notation "$\{P|d_T(P,A) = 3\}$" Is usually read:
    "the set of all points P such that the taxi distance from P to A is 3.

    c. Graph the set of all points P at Eclidean distance 3 from A ; that is, graph $\{P|d_E(P,A) = 3\}$

    Hopefully by now you have realized that in graphing $\{P|d_T(P,A) = 3\}$, you have created a taxi circle!

    d. In taxicab geometry, what is a reasonable numerical value for π?

5. Given $A = (-2,-1)$ and $B = (3,2)$. Graph the following sets of points.
(parts (a) through (e) are all asking you to graph a taxi circle, just using different notation each time)

    a. The taxi circle with center A and radius 2.
    b. $\{P|d_T(P,A) = 1\}$
    c. The set of all points P at a taxi distance 1½ from A.
    d. The taxi circle with center B and radius 4.
    e. $\left\{P|d_T(P,A) = 2\frac{1}{2}\right\}$

6. Given $A = (-2,-1)$ and $B = (3,2)$,
    a. Calculate $d_T(A,B)$.

    Now, complete b-h together, all on the **same graph.**

    b. Graph $\{P|d_T(P,A) = 3 \ \ and \ \ d_T(P,B) = 5\}$

    NOTE: when we use 'and' in this context we are looking for an *intersection*. Both $\{P|d_T(P,A) = 3\}$ and $\{ d_T(P,B) = 5\}$ by themselves are taxi circles. Graph them separately and then mark the points where they intersect. Do the same for (c) through (g). Part (h) combines all the previous parts.

    c. Graph $\{P|d_T(P,A) = 1 \ \ and \ \ d_T(P,B) = 7\}$
    d. Graph $\{P|d_T(P,A) = 0 \ \ and \ \ d_T(P,B) = 8\}$
    e. Graph $\left\{P|d_T(P,A) = 1\frac{1}{2} \ \ and \ \ d_T(P,B) = 6\frac{1}{2}\right\}$
    f. Graph $\{P|d_T(P,A) = 4 \ \ and \ \ d_T(P,B) = 4\}$
    g. Graph $\{P|d_T(P,A) = 5 \ \ and \ \ d_T(P,B) = 3\}$
    h. Graph $\{P|d_T(P,A) + \ d_T(P,B) = d_T(A,B)\}$

7. Given $A = (-7, -3)$ and $B = (5,2)$,

    a. Calculate $d_E(A, B)$
    b. Graph $\{P | d_E(P, A) + \ d_E(P, B) = d_E(A, B)\}$

    NOTE: If you model this problem after what you did for problem 6, a compass will be helpful.

8. On a sheet of graph paper mark each pair of points A and B and then graph $\{P | d_T(P, A) + \ d_T(P, B) = d_T(A, B)\}$ for the following A and B.

    a. $A = (-2,3)$ and $B = (1, -4)$
    b. $A = (1, -3)$ and $B = (4,0)$
    c. $A = (2,1)$ and $B = (6,1)$
    d. $A = (1,1)$ and $B = (1,4)$

9. Given $A = (-2, -1)$ and $B = (3,2)$

    a. Graph $\{P | d_T(P, A) = 5 \ \ and \ \ d_T(P, B) = 5\}$
    b. Graph $\{P | d_T(P, A) = 7 \ \ and \ \ d_T(P, B) = 7\}$
    c. Graph $\{P | d_T(P, A) = 4 \ \ and \ \ d_T(P, B) = 4\}$
    d. Graph $\{P | d_T(P, A) = \ \ d_T(P, B)\}$

10. Repeat Exercise 9 using d_E in place of d_T. **(A compass will be helpful.)**

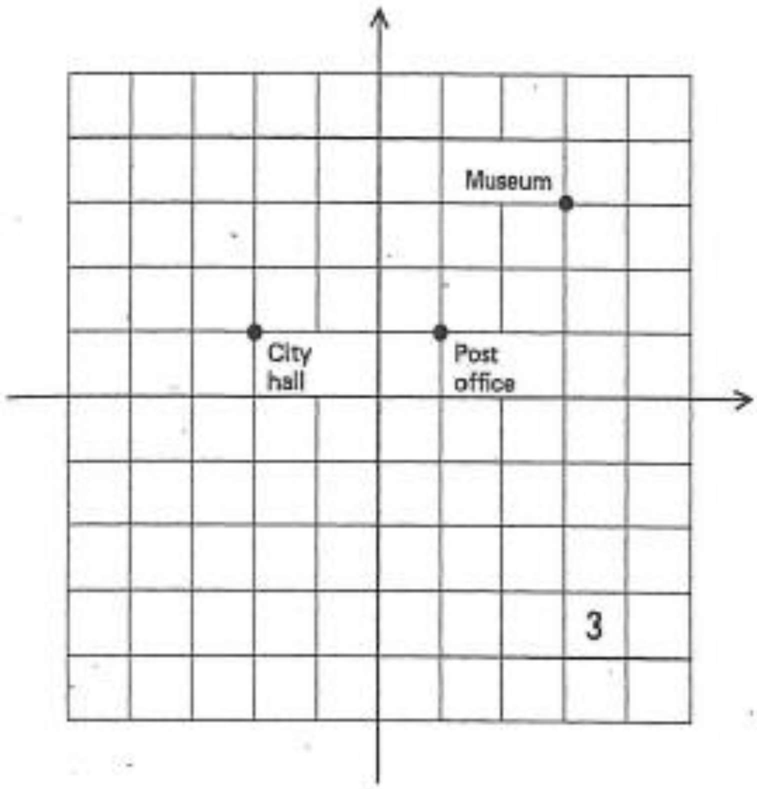
11. For each pair of points A and B, graph $\{P | d_T(P, A) = \ \ d_T(P, B)\}$
    a. $A = (0,0)$ and $B = (4,2)$
    b. $A = (0,0)$ and $B = (2,4)$
    c. $A = (0,0)$ and $B = (3,3)$ (Watch out!)
    d. $A = (-1,1)$ and $B = (4,1)$

12. Plot $A = (-3,0)$ and $B = (1,2)$ and then graph $\{P | d_T(P, A) = \ \ 2 * d_T(P, B)\}$.

13. Repeat Exercise 12 using d_E in place of d_T.

SOME APPLICATIONS

Taxicab Geometry is a more useful model of urban geography than is Euclidean geometry. Only a pigeon would benefit from the knowledge that the Euclidean distance from the Post Office to the Museum (Fig 3) is $\sqrt{8}$ blocks while the Euclidean distance from the Post Office to the City Hall is $\sqrt{9} = 3$ blocks. This information is worse than useless for a person who is constrained to travel along streets or sidewalks. For people, taxicab distance is the "real" distance. It is not true, for people, that the Museum is "closer to the Post Office than the City Hall is. In fact, just the opposite is true. (What are the two taxicab distances?)

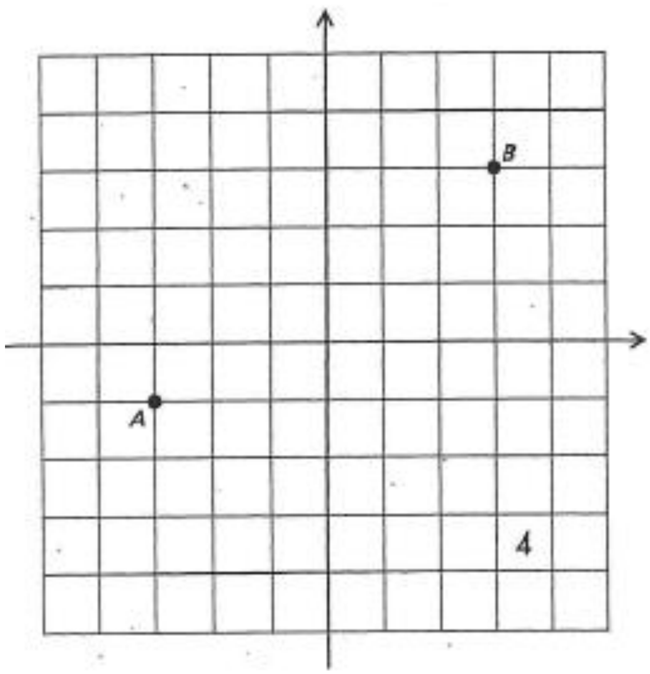


While taxicab geometry is a better mathematical model of urban geography than is Euclidean geometry, it is not perfect. Many simplifying assumptions have been made about the city. All the streets are assumed to run straight north and south or straight east and west; streets are assumed to have no width; buildings are assumed to be of point size… You should not be greatly disturbed by these assumptions. True, no city is exactly like the ideal one we have in mind. Still many parts of many cities are not too different from it. The things we learn about our ideal model will have application in certain real urban situations.

The process of setting up a mathematical model of a real situation nearly always involves making simplifying assumptions. Without them the mathematical problems tend to be too involved and difficult to solve, or even to set up. Later, you will have the opportunity to see some of the mathematical complications that arise when we alter our ideal model to make it more realistic.

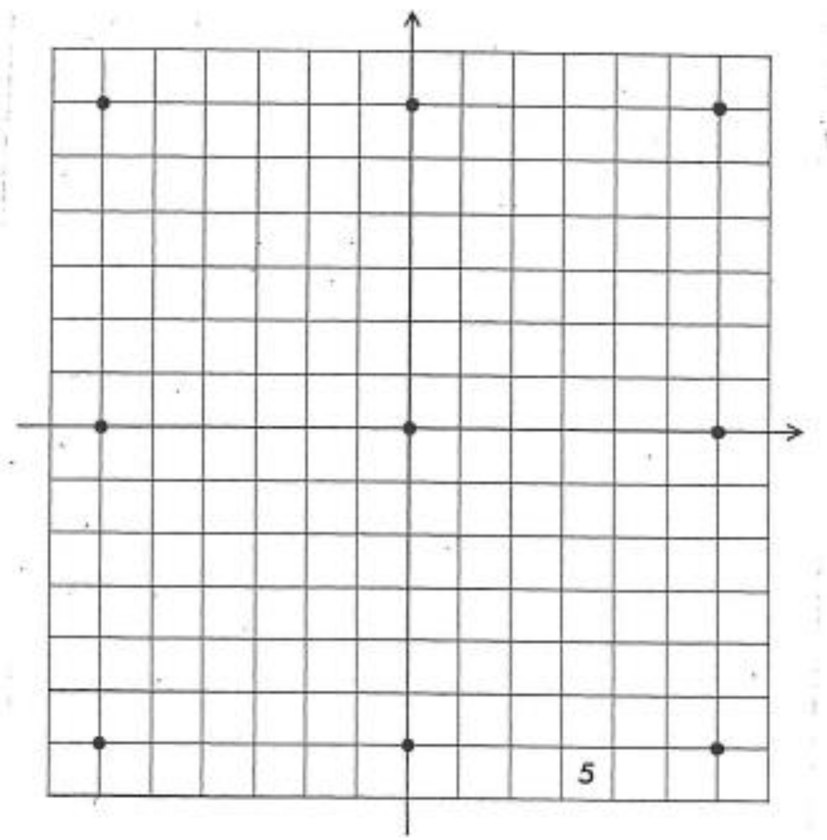
PROBLEM SET 2

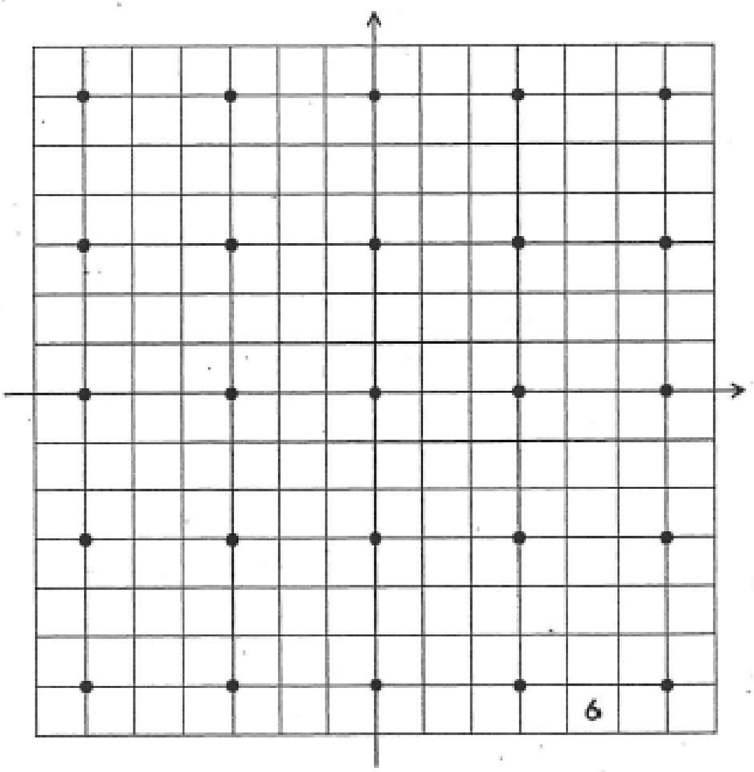
1. List some additional simplifying assumptions that we have made about Ideal City.
2. Alice and Bruno are looking for an apartment in Ideal City. Alice works as an acrobat at amusement park $A = (-3, -1)$. Bruno works as a bread taster in bakery $B = (3,3)$. (See Fig. 4.) Being ecologically aware, they walk wherever they go. They have decided their apartment should be located so that the distance Alice has to walk to work plus the distance Bruno has to walk to work is as small as possible. Where should they look for an apartment?



3. In a moment of chivalry, Bruno decides that the sum of the distances should still be a minimum, but Alice should not have to walk any farther than he does now where could they look for an apartment?
4. Alice agrees that the sum of the distances should be a minimum, but she is adamant that they both have exactly the same distance to walk to work. Now where could they live?
5. After a day of fruitless apartment hunting they decide to widen their area of search. The only requirement they keep is that they both be the same distance from their jobs. Now where should they look?
6. After another luckless day they finally agree that all that really matters is that Bruno be closer to his job than Alice is to hers. Now where can they look?
7. The dispatcher for the Ideal City Police Department receives a report of an accident at $X = (-1,4)$. There are two police cars in the area, car C at $(2,1)$ and car D at $(-1,-1)$. Which car should she send to the scene of the accident?
8. A builder wants to put up an apartment building within six blocks of the shopping center $S = (-3,0)$ and within four blocks of the tennis courts $T = (2,2)$. Where can he build?

9. The newly elected mayor has promised to install drinking fountains in Ideal City so that every citizen is within three blocks of a drink of water. She discovers that money for civic improvements is rather scarce. Her three aides present her with three plans for locating fountains (Figs. 5,6, and 7). Which should she probably pick, and why?
10. The telephone company wants to set up pay-phone booths so that everyone living within twelve blocks of the center of town is within four blocks of a pay phone. How few booths can they get by with, and where should they be located?
11. A group of students has decided to start a Junior Achievement business of custom finishing furniture. They will buy unfinished furniture at warehouse $W = (-3,2)$, transport it to their shop S for finishing, and then deliver it to retail store $R = (5,-1)$ for sale. Where should they locate their shop S if they want to minimize the distance they will have to haul furniture?
12. There are three high schools in Ideal City: Fillmore at $(-4,3)$, Grant at $(2,1)$, and Harding at $(-1,-6)$. Draw in school district boundary lines so that each student in Ideal City attends the high school nearest his home.
13. If Burger Baron wants to open a hamburger stand equally distance from each of the three high schools, where should it be located?
14. A fourth high school, Polk High, has just been built at $(2,5)$. Redraw the school-district boundary lines.





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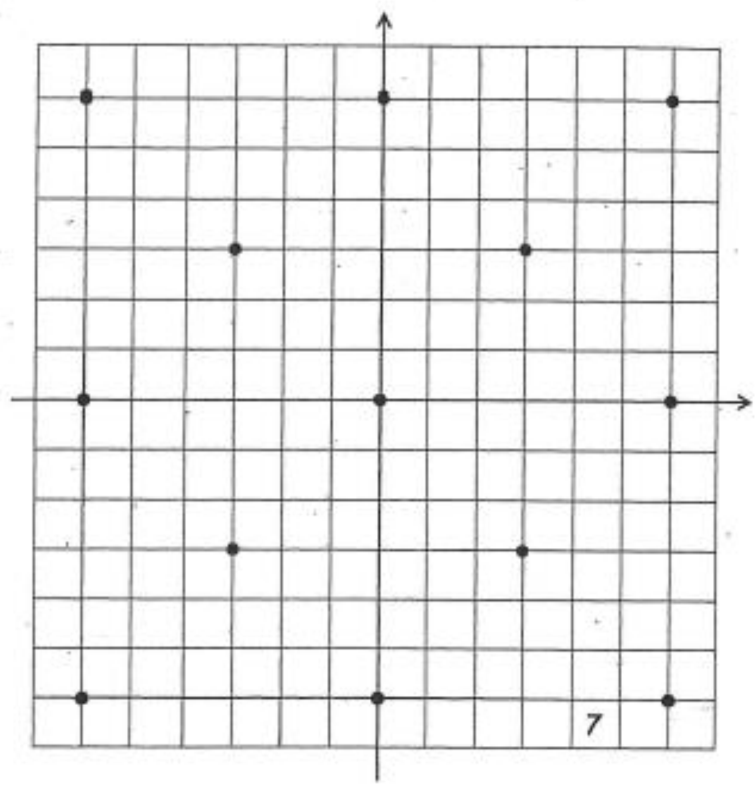

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SECTION II: BUILDING INDEPENDENCE

In this section, the problems will start out directed and you will slowly need to make more decisions for yourself. However, the decisions you are asked to make will be similar to problems you have already done. Use the steps given to you and do not reinvent the wheel unnecessarily.

Some Geometric Figures

We have already seen how some familiar geometric figures are transmuted in taxicab geometry. For example, circles in taxicab geometry are squares. As another example, the set of all points equidistant from two given points A and B looks quite different in taxicab geometry than in Euclidean geometry. In Euclidean geometry it is just the perpendicular bisector of $\overline{AB}$. In taxicab geometry it can have a variety of shapes (See Exercise 11 of Section 1), but only rarely turns out to be the perpendicular bisector of $\overline{AB}$.

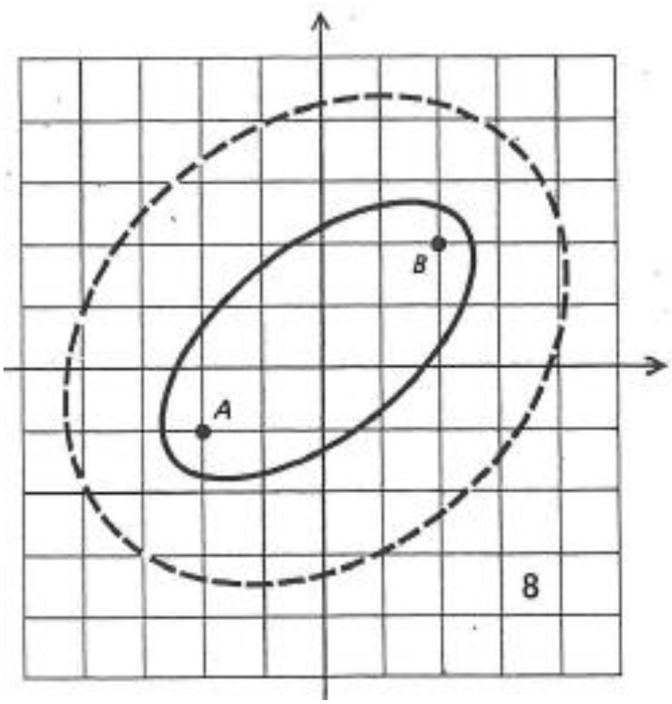
There are other useful geometric figures which can be defined in terms of distance and which deserve study in both Euclidean and taxicab geometry. One such figure is the ellipse. By definition, an ellipse is the set of all points the sum of whose distances from two given points is a constant. If we let $A = (-2, -1)$ and $B = (2,2)$ be the two given points called the foci of the ellipse, then one Euclidean ellipse with foci A and B is

$\{P | d_E(P, A) + d_E(P, B) = 6\}$.

This ellipse is the solid one in Fig. 8. Another Euclidean ellipse with foci A and B is

$\{P | d_E(P, A) + d_E(P, B) = 9\}$.

It is the dotted one in Fig. 8.



A procedure for sketching an ellipse, for example the solid one $\{P|d_E(P,A)+d_E(P,B)=6\}$ with foci $A=(-2,-1)$ and $B=(2,2)$, is as follows.

(As you read the following steps, you could either just look at Fig. 9 as you read or you could try and follow the directions yourself and use Fig. 9 to check your work.)

i) Use a compass to draw a circle of radius 4 with center A and a circle of radius 2 with center B. (These circles are the solid ones in Fig. 9.) Any point lying on both these circles will be at distance 4 from A and 2 from B; thus the sum of its distances from A and B will be 6, and it will be a point of the ellipse we are after.

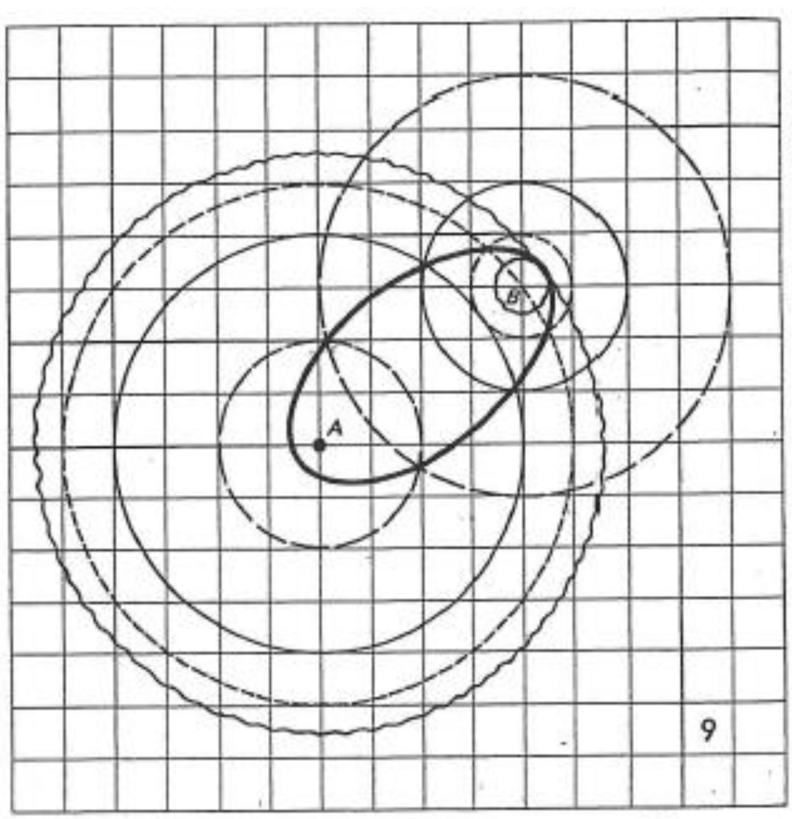
ii) Draw a circle of radius 2 with center A and a circle of radius 4 with center B. (These are the dashed ones in Fig. 9.) the intersection of these two circles contributes two more points to the ellipse.

iii) Draw a circle of radius 5 with center A and a circle of radius 1 with center B. (These are the dotted ones in Fig. 9.) Their intersection contributes two more points to the ellipse.

iv) Draw a circle of radius 5 ½ with center A and a circle of radius ½ with center B. (These are the wavy ones in Fig. 9.) note that the intersection of these two circles is a single point.

v) Draw other pairs of circles with centers A and B and radii that add up to 6, to find other points on the ellipse.

vi) When you have found enough points to see the general shape of the ellipse, join the points with a smooth curve.



PROBLEM SET 3

1. Mark $A = (-2,-1)$ and $B = (2,2)$ on a sheet of graph paper. Sketch the ellipse, $\{P|d_E(P,A) + d_E(P,B) = 9\}$, by following these steps, all on one graph.
    a. With a compass, draw lightly a circle with center A and and radius 4 ½, and another circle with center B and radius 4 ½. Darken their points of intersection.
    b. Repeat (a) using Center A, radius 5, and center B, radius 4.
    c. Repeat using Center A, radius 6, and center B, radius 3.
    d. Repeat using Center A, radius 6 ½, and center B, radius 2 ½.
    e. Repeat using Center A, radius 7, and center B, radius 2.
    f. Repeat using Center A, radius 8, and center B, radius 1.
    g. Sketch the desired ellipse.

2. Using $A = (-2,-1)$ and $B = (2,2)$ as foci, sketch the following sets **all on one graph.** Use a different color for each one. (An efficient way to plot all these figures is to begin by drawing a whole family of circles centered at A and having radii 1,2,…,8, and another such family with centers at B.)
    a. $\{P|d_E(P,A) + d_E(P,B) = 11\}$
    b. $\{P|d_E(P,A) + d_E(P,B) = 8\}$
    c. $\{P|d_E(P,A) + d_E(P,B) = 5\}$

3. Points A and B are the same as in Exercise 2.
    a. Calculate $d_E(A,B)$. Does this calculation shed any light on parts (d), (e), and (c) of Exercise 2?
    b. What do you suppose is the general shape of the ellipse, $\{P|d_E(P,A) + d_E(P,B) = 100\}$?
    c. Kepler's First Law of planetary motion states that the orbit of each planet is an ellipse with the sun at one focus. The Earth's "other" focus is about 5 million kilometers from the sun. The sum of the Earth's distances from its foci is about 300 million kilometers. What is the general shape of the Earth's orbit?

4. Again mark $A = (-2,-1)$ and $B = (2,2)$ on a sheet of graph paper. Devise a procedure (modeled after the procedure for Euclidean ellipse) and sketch the taxicab ellipse: $\{P|d_T(P,A) + d_T(P,B) = 9\}$.

5. On a new sheet of graph paper, again mark $A = (-2,-1)$ and $B = (2,2)$, and copy the taxicab ellipse of Exercise 4. Now, on the same graph, sketch in different colors these other sets.
    a. $\{P|d_T(P,A) + d_T(P,B) = 13\}$
    b. $\{P|d_T(P,A) + d_T(P,B) = 7\}$
    c. $\{P|d_T(P,A) + d_T(P,B) < 13\}$

6. Sketch the taxicab ellipse, $\{P|d_T(P,A)+d_T(P,B)=10\}$, where $M=(-2,1)$ and $N=(4,1)$.

7. While Euclidean ellipses have their applications in the heavens (planetary orbits), taxicab ellipses have theirs on Earth. Alice and Bruno still have not found an apartment. Remember that Alice works at $(-3,-1)$ and Bruno works at $(3,3)$. They have now decided that the sum of the distances that they have to walk to work should be no more than 14 blocks. Where can they look for an apartment?

8. Ajax Industrial Corporation wants to build a factory in Ideal City in a location where the sum of its distances from the railroad station $C=(-5,-3)$ and the airport $D=(5,-1)$ is at most 16 blocks. For noise-control purposes, a city ordinance forbids the location of any factory within 3 blocks of the public library $L=(-4,2)$. Where can Ajax build?

9. On a sheet of graph paper mark $A=(-2,-1)$ and $B=(2,2)$.
   a. Devise a procedure (using your compass) and sketch this figure:
      $\{P|d_E(P,A)-d_E(P,B)=3\}$
   b. The figure you just sketched is one branch (one half) of a (Euclidean) hyperbola with foci A and B. The other branch of this hyperbola is
      $\{P|d_E(P,B)-d_E(P,A)=3\}$.
      Sketch it on the same graph.
   c. Explain why the set notation
      $\{P\mid |d_E(P,A)-d_E(P,B)|=3\}$
      describes the entire hyperbola (both branches).

10. On a new sheet of graph paper, again mark $A=(-2,-1)$ and $B=(2,2)$ and copy the hyperbola of Exercise 9. Now sketch these sets in different colors.
    a. $\{P|\,|d_E(P,A)-d_E(P,B)|=1\}$
    b. $\{P|\,|d_E(P,A)-d_E(P,B)|=0\}$
    c. $\{P|\,|d_E(P,A)-d_E(P,B)|=4\}$
    d. $\{P|\,|d_E(P,A)-d_E(P,B)|=5\}$
    e. $\{P|\,|d_E(P,A)-d_E(P,B)|=6\}$

11. Mark $A=(-3,1)$ and $B=(2,2)$ on a sheet of graph paper.
    Now devise a procedure and sketch the taxicab hyperbola
    $\{P|\,|d_T(P,A)-d_T(P,B)|=3\}$.

12. On a new sheet of graph paper again mark $A = (-3,1)$ and $B = (2,2)$ and copy the taxicab hyperbola of Exercise 11. Using a different color for each one, sketch the following additional figures.
    a. $\{P \mid |d_T(P,A) - d_T(P,B)| = 0\}$
    b. $\{P \mid |d_T(P,A) - d_T(P,B)| = 2\}$
    c. $\{P \mid |d_T(P,A) - d_T(P,B)| = 6\}$ (Be careful!)
    d. $\{P \mid |d_T(P,A) - d_T(P,B)| = 7\}$ (???)
    e. What is significant about the number 6?

13. Investigate the family of taxicab hyperbolas with foci $A = (-3,1)$ and $B = (5,1)$.

14. Investigate the family of taxicab hyperbolas with foci $A = (0,0)$ and $B = (4,4)$.

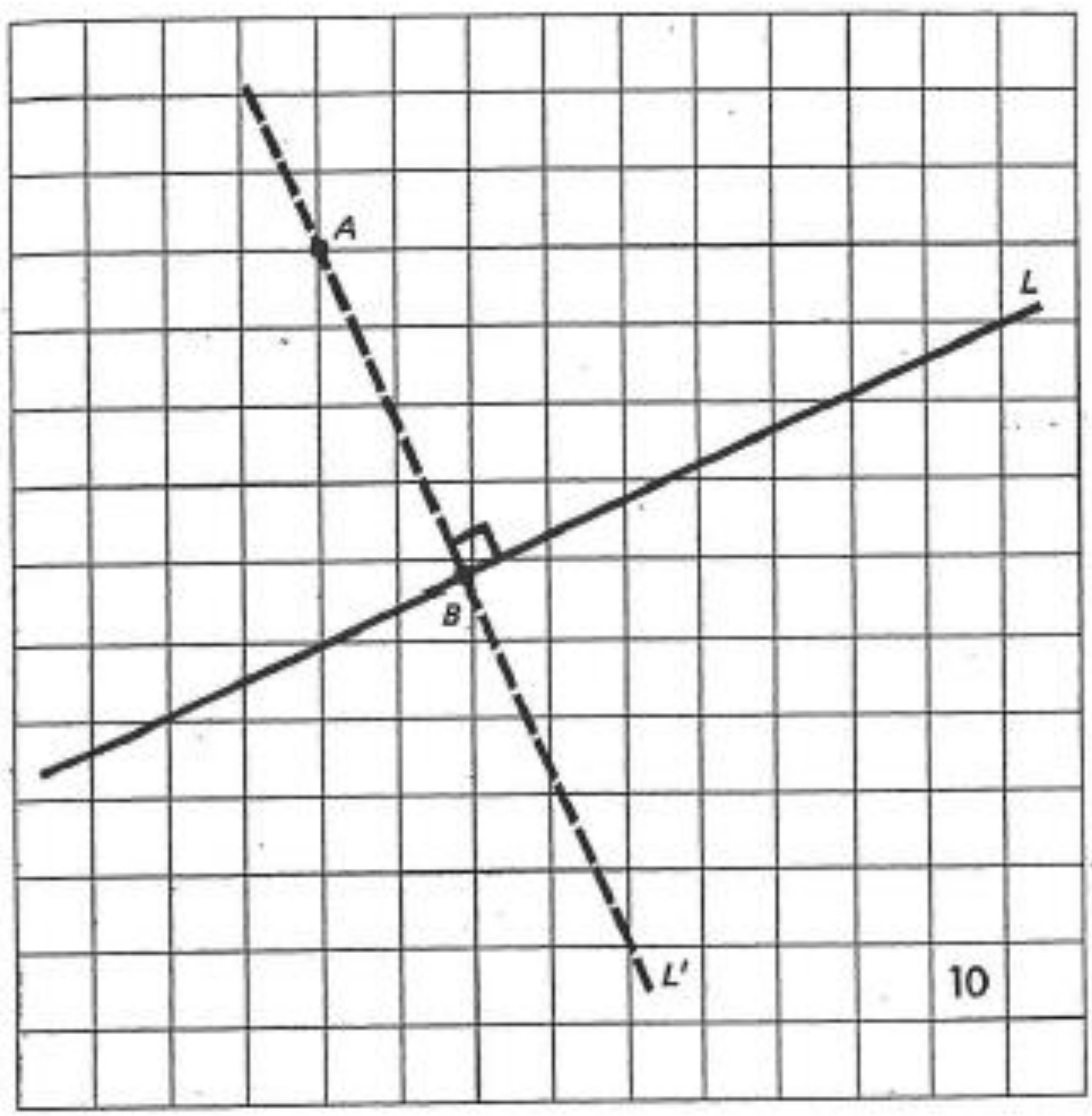
15. Alice and Bruno still don't have an apartment. Their latest agreement is that neither person should have to walk more than 4 blocks farther to work than the other person. Where can they look?

DISTANCE FROM A POINT TO A LINE

In Euclidean geometry there is a standard method for determining the distance from a point A to a line L. (See Fig. 10.) One first locates the line L' through A perpendicular to L. Then, letting B be the point of intersection of L' and L, one observes that the Eulcidean distance from A to B. In symbols,

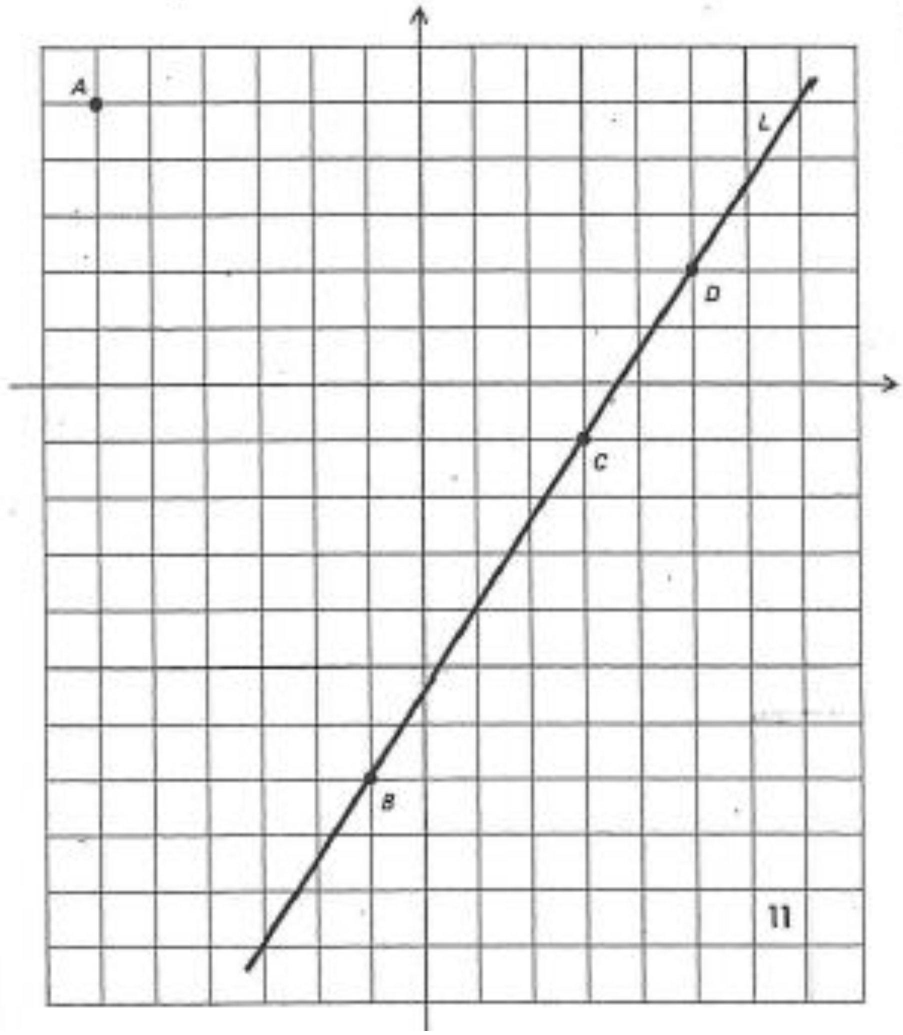
$$d_E(A, L) = d_E(A, B)$$

We shall see in the exercises that the procedure for finding the distance from a point to a line in taxicab geometry is quite different.



PROBLEM SET 4

1. Figure 11 shows a point A and a line L.
   a. Find $d_T(A, B)$
   b. Find $d_T(A, C)$
   c. Find $d_T(A, D)$
   d. Find a point P on L which is "as close as possible" to A in taxicab geometry. (Is P on the perpendicular to L through A?)
   e. What would you say is the taxicab distance from A to L?

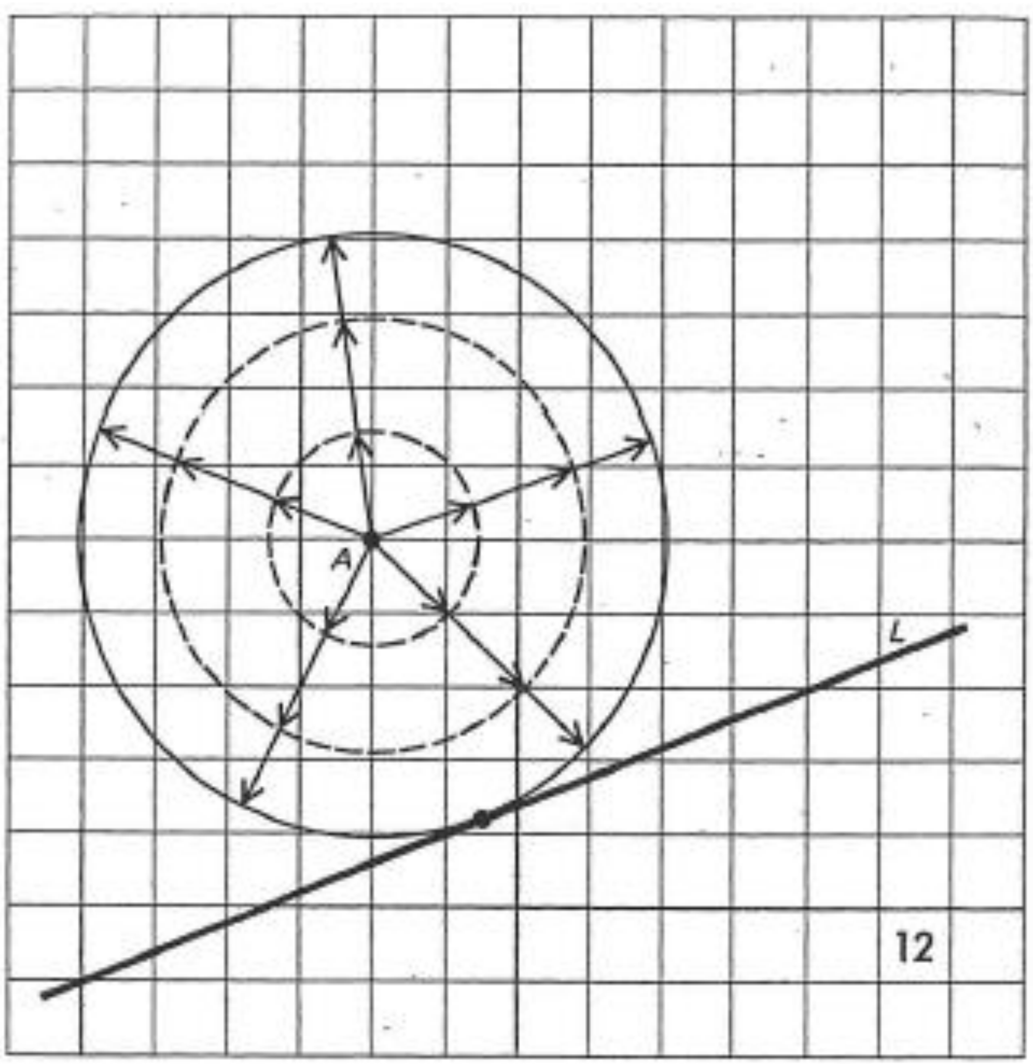


2. On a sheet of graph paper, sketch the point $A = (-3,2)$ and the line L passing through $(-6, -2)$ and $(0,0)$.
   a. Locate a point P on L that is as close as possible to A in taxicab geometry.
   b. Calculate $d_T(A, L)$.

3. Repeat Exercise 2 for $A = (-3,2)$ and L the line through $(-2, -1)$ and $(2,3)$.

4. So far we have considered procedures for finding the distance from a point to a line in Euclidean geometry and in taxicab geometry. But as yet we have formulated no definitions of what is meant by the distance from a point to a line. Exercises 1 through 3 suggest the following definition for taxicab geometry:
   $d_T(A, L) = The\ smallest\ of\ all\ the\ d_T(A, P)\ where\ P \in L.$
   This is often abbreviated
   $d_T(A, L) = min_{P \in L} d_T(A, P).$
   State the corresponding definition of $d_E(A, L)$.

5. Another way of thinking about how to find $d_E(A, L)$ is suggested by Fig. 12. Think of slowly inflating a circle with center A until it just touches L. Its radius at that moment is $d_E(A, L)$. Remembering what circles look like in taxicab geometry, reconsider Exercises 1(d), 1(e), 2, and 3 from the point of view of inflating taxicab circles. Would you change any answers?



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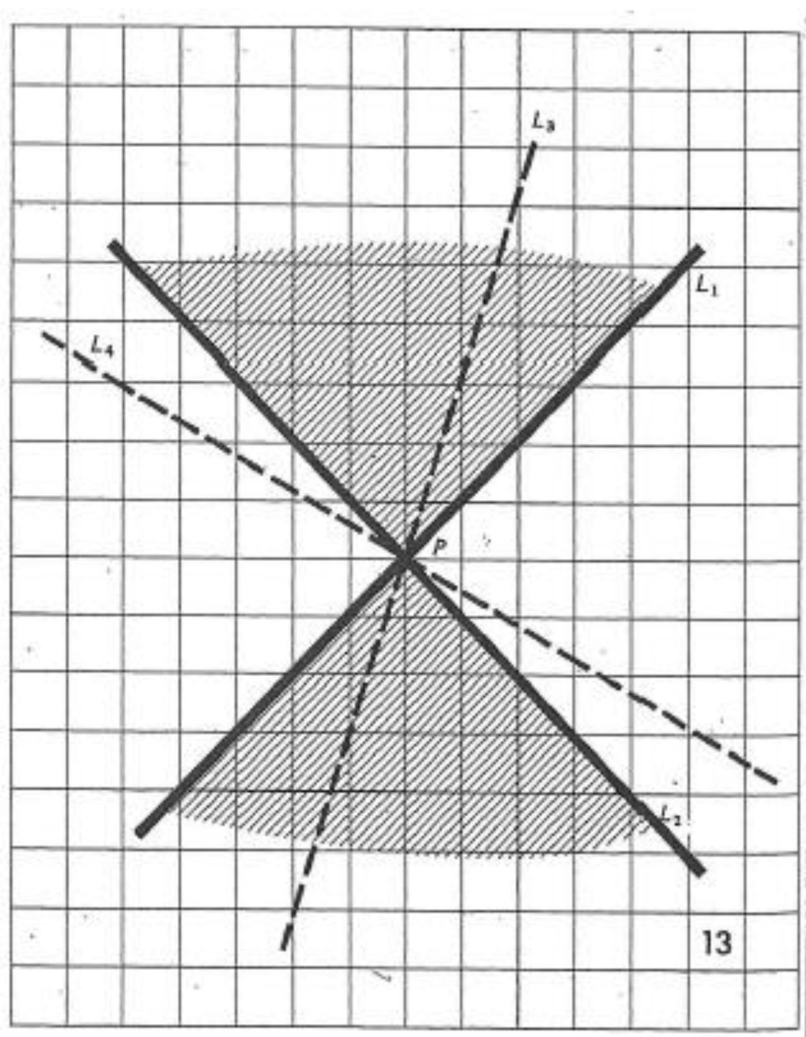
6. The Euclidean method for finding $d_E(A, L)$ is often stated loosely as follows. "Measure the distance from A to L along the *perpendicular*." We would like to formulate a similar loose, easily remembered verbalization for how to find $d_T(A, L)$. To do so, it is convenient to introduce some terminology. For any point P in the coordinate plane, we classify the lines through P under three headings. (See Fig. 13.)
    a. "45° lines" - L_1 and L_2 are the only 45° lines (through P).
    b. "Steep lines" – Any line (through P) that lies in the shaded region. L_3 is an example of a steep line.
    c. "gradual lines" – Any line (through P) that lies in the unshaded region. L_4 is an example of a gradual line.

   Complete these statements that tell how to determine the taxicab distance from a point A to a line L.
    a. "If L is steep, measure the distance from A to L along ___________________."
    b. "If L is gradual, ______________________."
    c. "If L is a 45° line, ____________________ ."

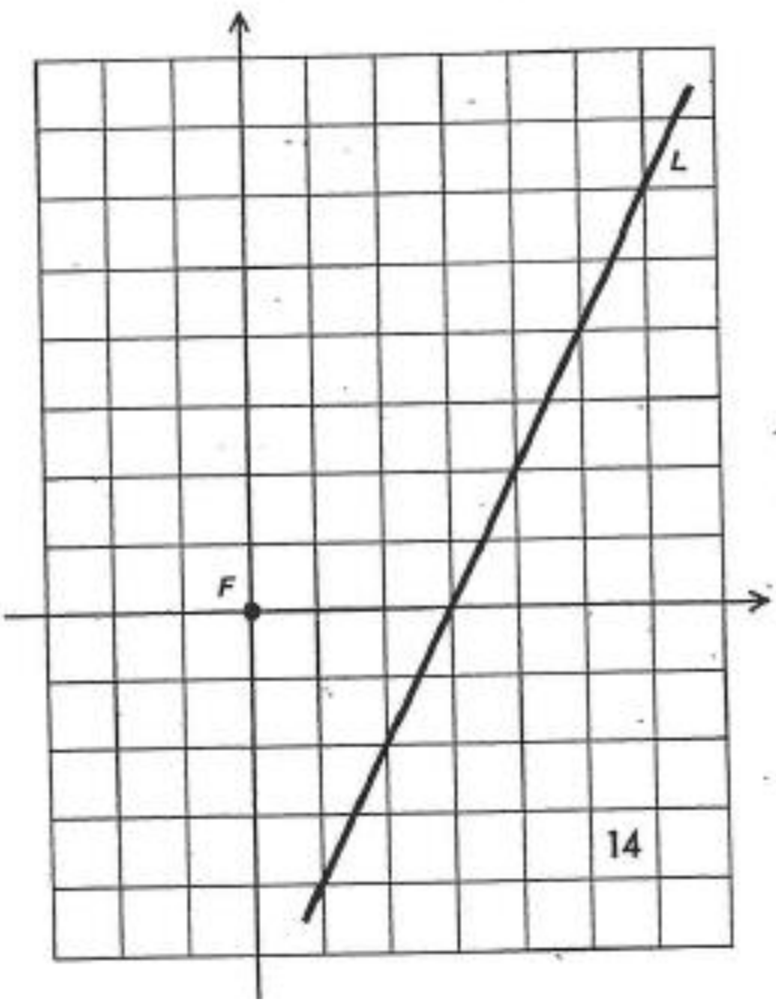


7. On a sheet of graph paper, draw the line L through $(3,0)$ and $(1,-4)$. Now,
   a. Sketch $\{P|d_T(P,L) = 2\}$
   b. Sketch $\{P|d_T(P,L) = 4\}$
   On the same sheet of graph paper, but in a different color,
   c. Sketch $\{P|d_E(P,L) = 2\}$
   d. Sketch $\{P|d_E(P,L) = 4\}$

8. Repeat Exercise 7 using the line L through $(0,0)$ and $(3,1)$.

9. Can you think of a line L for which
   $\{P|d_T(P,L) = 2\} = \{P|d_E(P,L) = 2\}$?

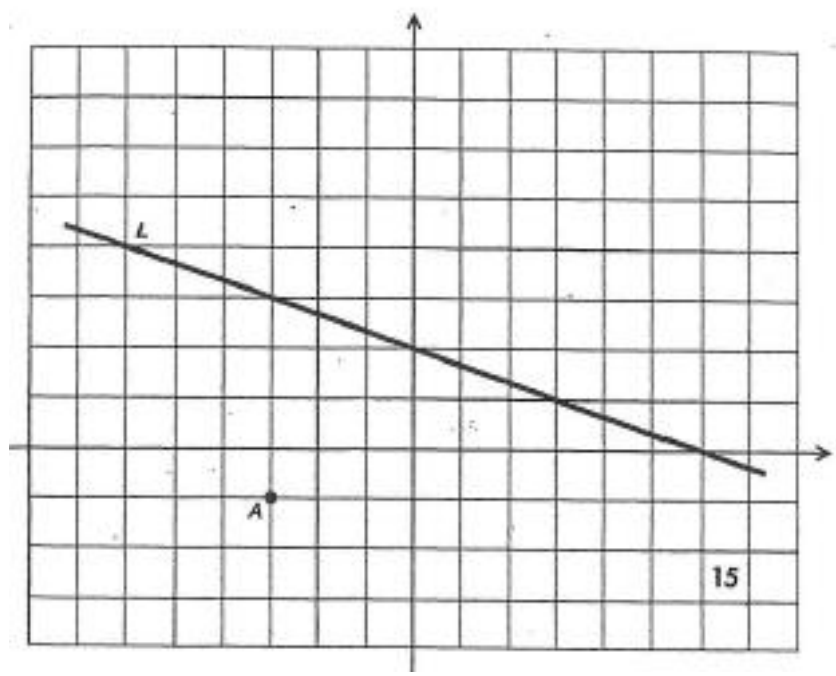
10. Figure 14 shows a point F and a line L.
   a. Sketch $\{P|d_T(P,F) = 2\}$
   b. Sketch $\{P|d_T(P,L) = 2\}$
   c. Sketch $\{P|d_T(P,F) = 2\}$ and $\{P|d_T(P,L) = 2\}$
   d. Sketch $\{P|d_T(P,F) = d_T(P,L)\}$



11. With F and L as in Exercise 10, sketch
$\{P|d_E(P,F) = d_E(P,L)\}$.
(A compass and ruler will be useful.)

12. The figure in Exercise 11 is known as a (Euclidean) *parabola*. F is called is *focus*, L its *directrix*. Thus we shall refer to
$\{P|d_T(P,F) = d_T(P,L)\}$
as the *taxi parabola* with focus F and directrix L. Devise a plan and sketch the taxi parabola with the focus F and directrix L given:
a. $F = (-2,2)$ and L is the line through $(-2,-2)$ and $(2,2)$;
b. $F = (0,4)$ and L is the line through $(0,0)$ and $(2,0)$.

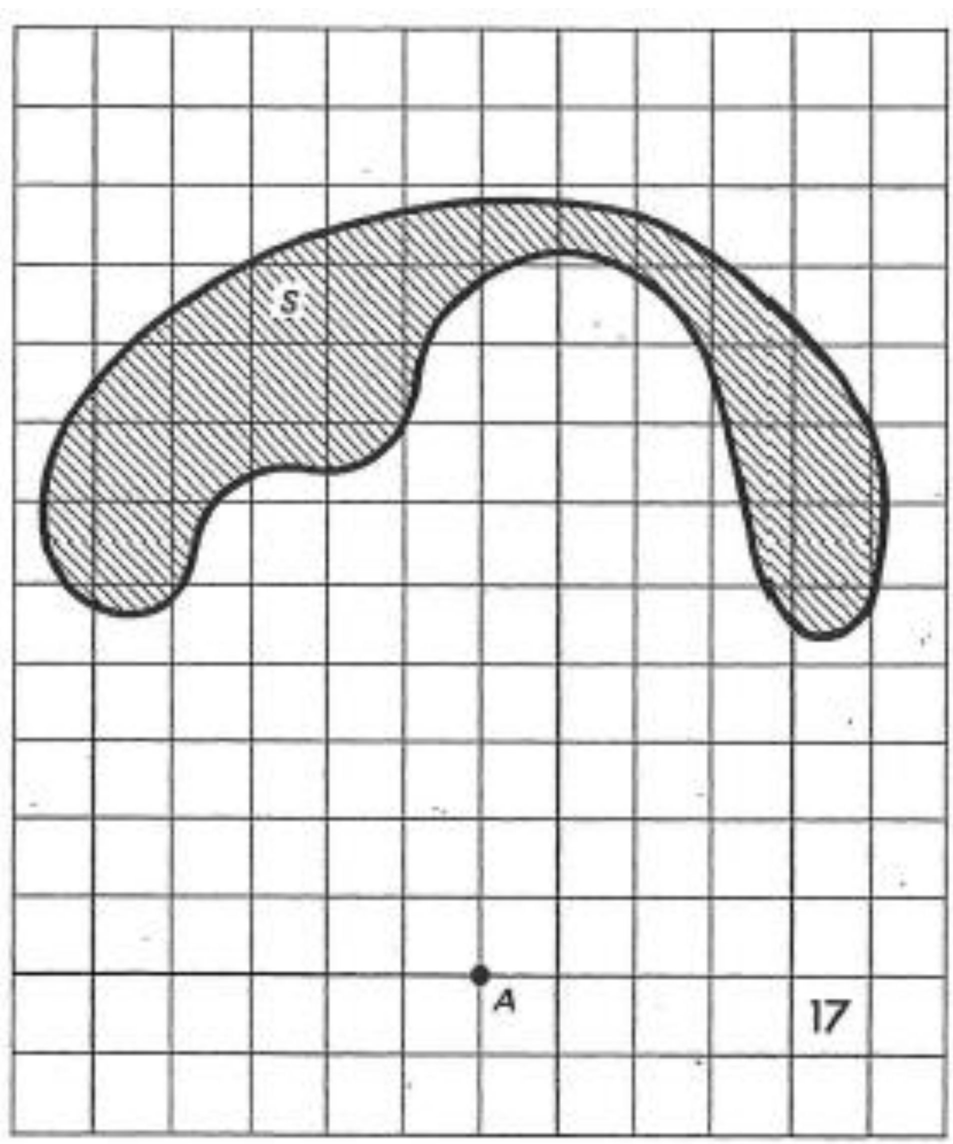
13. Alice still works as an acrobat at $A = (-3,-1)$, but Bruno has a new job as a conductor on the new mass-transit vehicle which runs along the line L shown in Fig. 15. One of Bruno's fringe benefits is that when he comes to work he can get on the vehicle at the point nearest his home. This sends Alice and Bruno off on another apartment search
    a. They want to live where the distance Alice has to walk to work plus the distance Bruno has to walk to work is a minimum. Where should they look?
    b. They change their minds and decide to live where they both have the same distance to walk to work. Where should they look?
    c. Where should they look if all that matters is that Alice have a shorter distance to walk than Bruno?
    d. Where should they look if they both want to be within 6 blocks of their job?
    e. Where should they look if the sum of the distances they have to walk is to be at most 6 blocks?



14. For old times' sake, Alice wants to walk from the amusement park $A = (-3, -1)$ to the bakery $B = (3,3)$, but she would like to stop along the way to watch the freight train go by on the track shown in Fig. 16. Where should she stop to watch the train if she wants to minimize the distance she walks? How long will her hike from A to B be?

15. Repeat Exercise 14 in the case where the (straight) railroad tracks pass through $(0, -8)$ and $(4,0)$.

16. In Fig. 17 a point A and a set S are shown. Approximate these two distances.
    a. $d_E(A, S)$
    b. $d_T(A, S)$



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SECTION III: INDEPENDENT EXPLORATION

In this section, you will get to choose from a variety of further explorations into taxicab geometry. You will devise and try your own plan.

We have investigated only a few of the questions suggested by taxicab geometry. Probably many others occurred to you as you worked through the preceding sections. It is likely that some of your questions have never been answered, or even asked, before these questions which you formulate yourself provide an ideal basis for truly original research. We encourage you to follow where your questions lead and to report on your discoveries. In mathematics, curiosity is the mother of invention.

Mathematical questions tend to spring from two main sources: the real world, and the abstract world of mathematics. The real world, with its city streets, provide much of the motivation for studying taxicab geometry in the first place. Questions of optimum locations for apartments, factories, phone booths, etc., produced a host of geometrical problems. The real world also suggests modifying our model from taxicab geometry to mass-transit geometry (see below), and this generated another whole array of mathematical questions which we have not yet investigated. (Quite a few new problems will arise if you try to do for, say, mass-transit geometry everything that we did for taxicab geometry.) Posing new "real" problems will lead to more geometrical questions about both taxicab and mass transit geometry.

Entire new geometries are also suggested by real-world cities. Change the route of the mass transit. Put a kink in it put in two mass transit lines. Put a pond somewhere in the city. Note that cities have a vertical dimension, and study taxicab geometry in space. The list of questions suggested by the real world seems to be endless. (some hints of these ideas showed up in problems at the end of Section II)

Purely mathematical considerations can also suggest further questions for study. For example, we might observe that the whole field of taxicab geometry was opened up by simply replacing the Euclidean distance function d_E by the taxicab distance function d_T. Recall that if $A = (a_1, a_2)$ and $B = (b_1, b_2)$, then the precise definitions of d_E and d_T were:

$$d_E(A,B) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}\ ,$$

$$d_T(A,B) = |a_1 - b_1| + |a_2 - b_2|.$$

A natural mathematical question to ask is this: What happens to the geometry if we replace d_E by some other distance function which need not have any connection with the real world?

The following page lists a series of possibilities for what you could choose to tackle in this last section. Whatever question you choose to pursue, make sure to write down a plan, and evaluate how well your plan worked once you have tried it.

EXPLORATION IDEAS

(1) Define taxi trig functions via wrapping of the unit taxi circle, and investigate their graphs.

(2) Define taxi triangles and investigate their graphs.

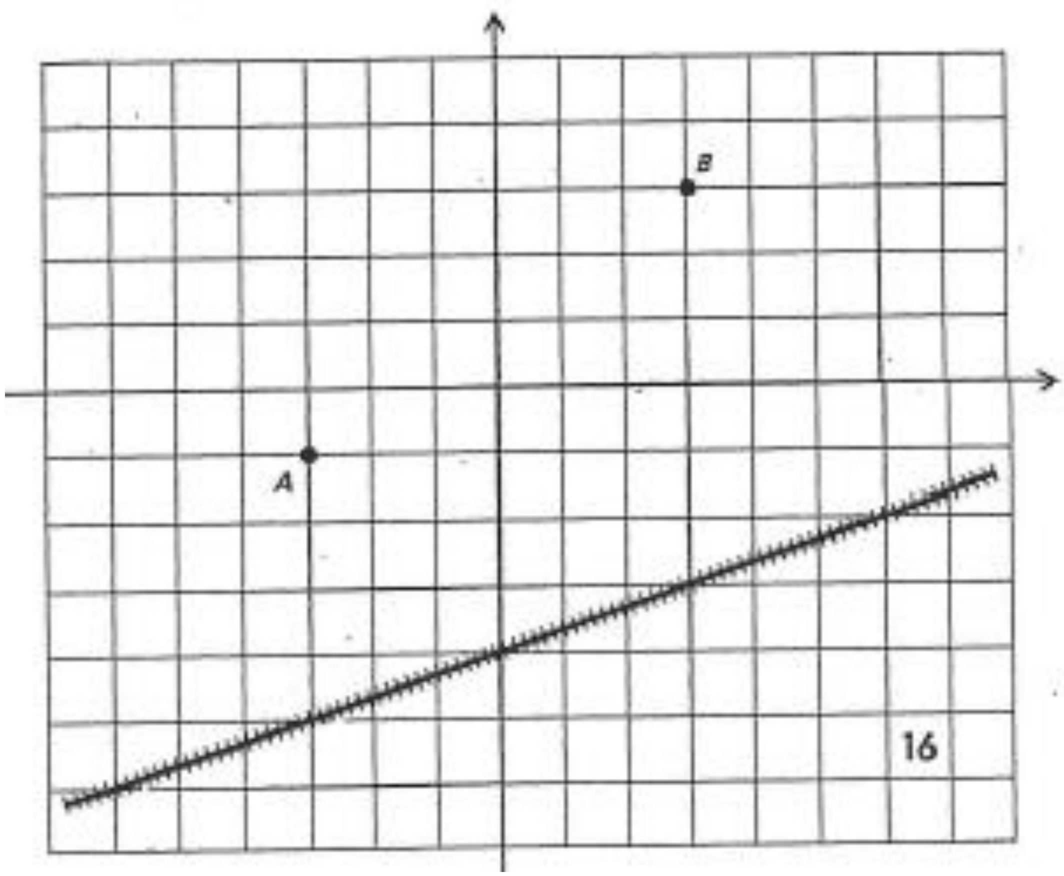
(3) Add some complexity to your taxi 'city,' such as a pond/lake, a mass transit line (see Fig. 16), or a third dimension and explore how things change.

(4) Instead of beginning with a square grid of streets as in Fig. 3, begin with an equilateral triangle grid and develop an entire "Chinese Checkers Geometry" parallel to Taxicab Geometry.

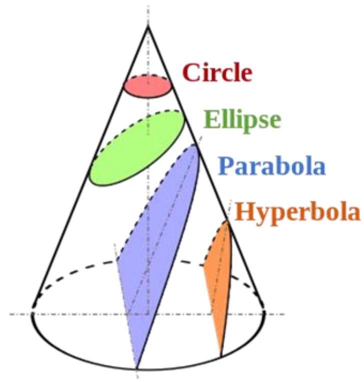
(5) Make up a new definition for the distance between two points, different from both Euclidean and taxi definitions. Then choose a couple of geometric ideas and explore how they would change.

(6) Is there another definition you think you could change and explore? Write the Euclidean definition and your changed definition and pick a few discrete geometric ideas to explore and redefine.

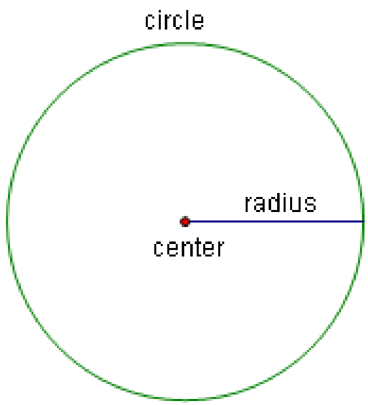
(7) Come up with your own idea for exploration and pursue it. Be sure to write out clearly what you would like to explore and how you plan to go about it.



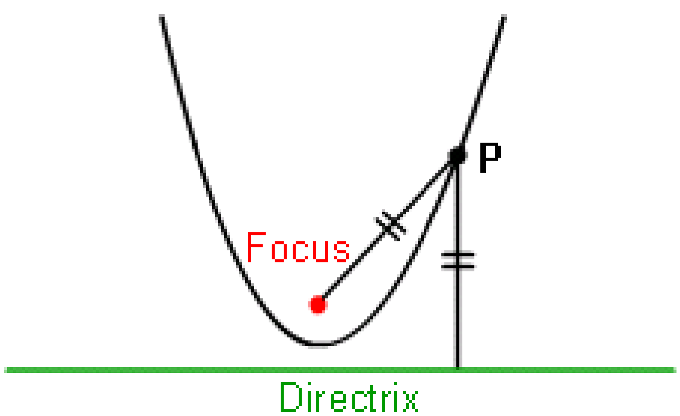
APPENDIX A: CONIC SECTIONS



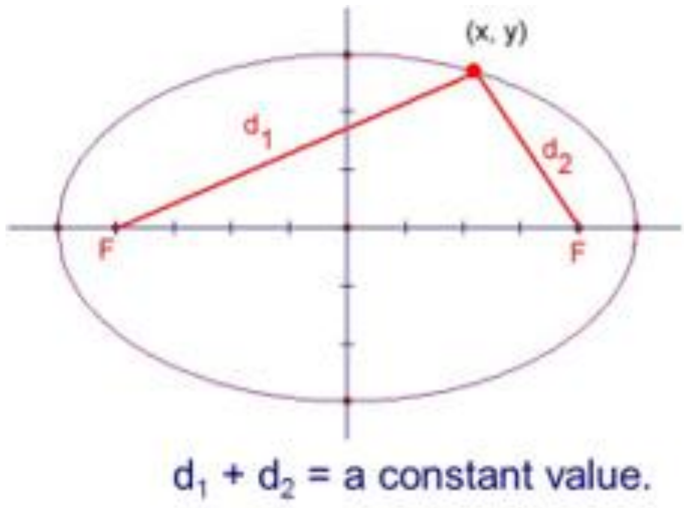
Circle Definition:
A circle is the set of all points in the plan which are equidistant (radius) from a point (center)



Parabola Definition:
A parabola is the set of all points in the plane which are equidistant from a point (focus) and a line (directrix).



Ellipse Definition:
An ellipse is the set of all points in the plane such that the sum of the distances from two points (foci) is a constant.



Hyperbola Definition:
A hyperbola is the set of all points in the plane such that the difference in the distances from two points (foci) is a constant.